$$\lambda_r=\lambda_0^{\,1} \qquad \varkappa_{r+1}=\varkappa_1^{\,1}$$
$$\lambda_{r+1}=\lambda_1^{\,1} \qquad \varkappa_{r+2}=\varkappa_2^{\,1}$$
$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$
$$\lambda_n=\lambda^1{}_{n-r} \qquad \varkappa_n=\varkappa^1{}_{n-r};\ \text{we get,}$$

$$\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \cdots\cdots (n-r)^{\lambda^1 n-r}\\ 0^m & \varkappa_1^{\,1}\ \varkappa_2^{\,1} \cdots\cdots \varkappa^1{}_{n-r}\end{pmatrix}=(-1)^{mr}\begin{pmatrix}r^{\lambda_0} (r+1)^{\lambda_1} \cdots\cdots n^{\lambda^1 n-r}\\ 0^m\ m^r \varkappa_1^{\,1}\ \varkappa_2^{\,1} \cdots\cdots \varkappa^1{}_{n-r}\end{pmatrix}.$$

Finally putting $n-r=n^1$, $n=n^1+r$, and dropping all accents,

$$\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \cdots\cdots n^{\lambda_n}\\ 0^m & \varkappa_1\ \varkappa_2 \cdots\cdots \varkappa_n\end{pmatrix}=(-1)^{mr}\begin{pmatrix}r^{\lambda_0} (r+1)^{\lambda_1} \cdots\cdots (n+r)^{\lambda_n}\\ 0^m\ m^r\ \varkappa_1\ \varkappa_1 \cdots\cdots \varkappa_n\end{pmatrix},$$

and this is the fourth formula of derivation.

5. The Four Formulas of Derivation.

We collect the four formulas of derivation. They are:

$$\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \cdots\cdots n^{\lambda_n}\\ 0^m & \varkappa_1\ \varkappa_2 \cdots\cdots \varkappa_n\end{pmatrix}=\begin{pmatrix}0^{\lambda_0+\lambda} & 1^{\lambda_1} \cdots\cdots n^{\lambda_n}\\ 0^{m+\lambda} & \varkappa_1\ \varkappa_2 \cdots\cdots \varkappa_n\end{pmatrix}$$

$$\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \cdots\cdots n^{\lambda_n}\\ 0^m & \varkappa_1\ \varkappa_2 \cdots\cdots \varkappa_n\end{pmatrix}=(-)^{n\kappa}\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \cdots\cdots n^{\lambda_n+\kappa}\\ 0^{m+\kappa} & (\varkappa_1+\varkappa)(\varkappa_2+\varkappa) \cdots\cdots (\varkappa_n+\varkappa)\end{pmatrix}$$

$$\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \cdots\cdots n^{\lambda_n}\\ 0^m & \varkappa_1\ \varkappa_2 \cdots\cdots \varkappa_n\end{pmatrix}=\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \cdots n^{\lambda_n} (n+1)^0 \cdots (n+\varkappa)^0\\ 0^m & \varkappa_1\ \varkappa_2 \cdots\cdots\cdots \varkappa_n\ 0^\kappa\end{pmatrix}$$

$$\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} \cdots\cdots n^{\lambda_n}\\ 0^m & \varkappa_1\ \varkappa_2 \cdots\cdots \varkappa_n\end{pmatrix}=(-1)^{mr}\begin{pmatrix}r^{\lambda_0} (r+1)^{\lambda_1} \cdots (n+r)^{\lambda_n}\\ 0^m m^r\ \varkappa_1\ \varkappa_2 \cdots\cdots \varkappa_n\end{pmatrix}.$$

These four kinds of derivation here correspond to the third, first, second, and fourth kinds of derivation, respectively, in the resultant theory.

F. THE CONJUGATE PRODUCT AND FUNCTION.

1. Definitions.

With respect to the product $(a_{r_1})^{p_1}(a_{r_2})^{p_2} \cdots (a_{r_\mu})^{p_\mu}$, the function

$$a_0{}^n \Sigma (a r_1)^{p_1+p_2+\cdots\cdots p_\mu} [\alpha(r_2-r_1)]^{p_2+p_3+\cdots\cdots p_\mu} [\alpha(r_3-r_2)]^{p_3+p_4+\cdots\cdots p_\mu} \cdots$$
$$[\alpha(r_\mu-r_{\mu-1})]^{p_\mu}$$
$$=a_0{}^n \Sigma (\alpha_1 \alpha_2 \cdots \alpha_r)^{p_1} (\alpha_1 \alpha_2 \cdots \alpha_{r_2})^{p_2} (\alpha_1 \alpha_2 \cdots \alpha_{r_3})^{p_3} \cdots (\alpha_1 \alpha_2 \cdots \alpha_{r_\mu})^{p_\mu},$$

is called the conjugate symmetric function, and with respect to the function, the product is called the conjugate product.

The number n is the order of the function and $n=p_1+p_2+\ldots.p_\mu$. If the function is given as

$$a_0^{\kappa_1}\Sigma\alpha_1^{\kappa_1}\alpha_2^{\kappa_2}\ldots\alpha_n^{\kappa_n}=a_0^{\kappa_1}\Sigma\alpha_1^{\kappa_1-\kappa_2}(\alpha_1\alpha_2)^{\kappa_2-\kappa_3}(\alpha_1\alpha_2\alpha_3)^{\kappa_3-\kappa_4}\ldots(\alpha_1\alpha_2\ldots.\alpha_n)^{\kappa_n},$$

it is clear that the conjugate product is,

$$a_1^{\kappa_1-\kappa_2}a_2^{\kappa_2-\kappa_3}a_3^{\kappa_3-\kappa_4}\ldots.a_n^{\kappa_n}.$$

2. The Coefficient of the Conjugate Product in a Symmetric Function. Completely Reducible Forms.

It will next be proved that the coefficient of the conjugate product in a symmetric function is completely reducible, and is equal to $(-1)^w$, where w is the weight of the function. According to 1, this coefficient may be written :

$$\begin{pmatrix}1^{\kappa_1-\kappa_2}\,2^{\kappa_2-\kappa_3}\ldots.(n-1)^{\,\kappa_{n-1}-\kappa_n}\,n^{\kappa_n}\\ 0^{\kappa_1}\;\kappa_1\;\kappa_2\;\ldots\ldots\ldots\ldots\ldots\kappa_n\end{pmatrix}.$$

By D, 2 (second reduction), we may reduce this, as follows :

$$\begin{pmatrix}1^{\kappa_1-\kappa_2}\,2^{\kappa_2-\kappa_3}\ldots.(n-1)^{\,\kappa_{n-1}-\kappa_n}\,n^{\kappa_n}\\ 0^{\kappa_1}\;\kappa_1\;\kappa_2\;\ldots\ldots\ldots\ldots\ldots\kappa_n\end{pmatrix}$$

$$=(-1)^{n\kappa_n}\begin{pmatrix}1^{\kappa_1-\kappa_2}\,2^{\kappa_2-\kappa_3}\ldots\ldots\ldots(n-1)^{\,\kappa_{n-1}-\kappa_n}\\ 0^{\kappa_1-\kappa_n}(\kappa_1-\kappa_n)(\kappa_2-\kappa_n)\ldots.(\kappa_{n-1}-\kappa_n)\end{pmatrix}$$

$$=(-1)^{n\kappa_n+(n-1)(\kappa_{n-1}-\kappa_n)}\begin{pmatrix}1^{\kappa_1-\kappa_2}\,2^{\kappa_2-\kappa_3}\ldots\ldots\ldots\ldots\ldots(n-2)^{\,\kappa_{n-2}-\kappa_{n-1}}\\ 0^{\kappa_1-\kappa_{n-1}}(\kappa_1-\kappa_{n-1})(\kappa_2-\kappa_{n-1})\ldots.(\kappa_{n-2}-\kappa_{n-1})\end{pmatrix}.$$

We have made here two reductions, one after the other. Proceeding in this way, we can now see that after n reductions of the same kind, we obtain,

$$\begin{pmatrix}1^{\kappa_1-\kappa_2}\,2^{\kappa_2-\kappa_3}\ldots\ldots n^{\kappa_n}\\ 0^{\kappa_1}\;\kappa_1\;\kappa_2\ldots\ldots\ldots\;\kappa_n\end{pmatrix}$$

$$=(-1)^{n\kappa_n+(n-1)(\kappa_{n-1}-\kappa_n)+(n-2)(\kappa_{n-2}-\kappa_{n-1})+\ldots.(\kappa_1-\kappa_2)\binom{1^0}{0^0}}$$

$$=(-1)^{\kappa_1+\kappa_2+\ldots.\kappa_n}\binom{1}{1}=(-1)^w.$$

This is the completely reducible form in the theory of symmetric functions. It may also be written as

$$\begin{pmatrix}0^{\lambda_0}\,1^{\lambda_1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_n}\\ 0^m(\lambda_1+\lambda_2+\ldots.\lambda_n)(\lambda_2+\lambda_3+\ldots.\lambda_n)(\lambda_3+\lambda_4+\ldots.\lambda_n)\ldots.(\lambda_{n-1}+\lambda_n)\lambda_n\end{pmatrix}.$$

3. Correspondence of the Conjugate Product with the Completely Reducible Term of $R_{m,n}$.

It will yet be shown more in detail that the conjugate product in a symmetric function corresponds to the completely reducible term in the resultant theory. According to the method of A, 5, the resultant of f and ϕ may be expressed as

$$\Sigma(-1)^{m\,n+g_1}(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_0)^m\Sigma(\beta p_1)^{m-r_1}(\beta p_2)^{m-r_2}\ldots(\beta p_\mu)^{m-r_\mu},$$

where g_1 is the weight of $(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}$. $[r_1, r_2 \ldots r_\mu$ are supposed to be arranged in the order of ascending magnitude]. A group of terms will be represented by

$$(-1)^{m\,n+g_1}(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_0)^m\Sigma(\beta p_1)^{m-r_1}\ldots(\beta p_\mu)^{m-r_\mu}.$$

Among these terms will be the one containing the conjugate product of the symmetric function involved. It will be

$$(-1)^{m\,n+g_1}(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_0)^m\frac{(-1)^{m\,n-g_1}}{b_0{}^{m-r_1}}(b_{p_1})^{r_2-r_1}(b_{p_1+p_2})^{r_3-r_1}$$

$$\ldots(b_{p_1+p_2+\ldots\ldots p_\mu})^{m-r_\mu}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_0)^{r_1}(b_{p_1})^{r_2-r_1}(b_{p_1+p_2})^{r_3-r_1}$$

$$\ldots(b_{p_1+p_2+\ldots\ldots p_\mu})^{m-r_\mu}.$$

But this by A, 1, (2) is the completely reducible term of $R_{m,n}$.

G. RECURRENCE FORMULA FOR THE COEFFICIENT

$$\binom{0^{\lambda_0}\,1^{\lambda_1}\,2^{\lambda_2}\ldots\ldots\ldots\ldots\ldots n^{\lambda_n}}{0^m m^{\mu_0}(m-1)^{\mu_1}(m-2)^{\mu_2}\ldots\ldots 0^{\mu_m}}.$$

1. Statement of Requirement.

Since all the coefficients of terms in symmetric functions can be reduced to such as are normal forms, or else are completely reducible and have the coefficient $(-1)^w$, we require farther only a means of calculating the normal forms. We will obtain a recurrence formula for the normal form

$$\binom{0^{\lambda_0}\,1^{\lambda_1}\ldots\ldots\ldots\ldots n^{\lambda_n}}{0^m m^{\mu_0}(m-1)^{\mu_1}\ldots\ldots 0^{\mu_m}},\ \text{where } m\geq n.$$

2. Derivation of the Formula.

From the equation $\phi=0$, we have

$$b_0\beta_1{}^m+b_1\beta_1{}^{m-1}+b_2\beta_1{}^{m-2}+\ldots+b_r\beta_1{}^{m-r}+\ldots\ldots+b_m\beta_1{}^{m-n}=0.$$

We multiply this equation by

$$(b_0)^{m-1}(\beta_2\beta_3\ldots\beta_{\mu_0})^m(\beta_{\mu_0+1}\ldots\beta_{\mu_0+\mu_1})^{m-1}\ldots(\beta_{\mu_{m-2}+1}\ldots\beta_{\mu_0+\mu_1+\ldots\mu_{m-1}})^1$$

$$(\beta_{\mu_{m-1}+1}\ldots\beta_{\mu_0+\ldots\mu_m})^0 \text{ and get } (b_0)^m(\beta\mu_0)^m(\beta\mu_1)^{m-1}(\beta\mu_2)^{m-2}\ldots.$$

$$(\beta\mu_m)^0+(b_0)^{m-1}b_1[\beta(\mu_0-1)]^m[\beta(\mu_1+1)]^{m-1}\ldots.(\beta\mu_m)^0+\ldots.$$

$$+(b_0)^{m-1}b_r[\beta(\mu_0-1)]^m(\beta\mu_1)^{m-1}(\beta\mu_2)^{m-2}\ldots.[\beta(\mu_r+1)]^{m-r}\ldots\ (\beta\mu_m)^0+\ldots=0.$$

We will next take the sum of all such equations, using all possible substitutions in the summation. We obtain a series of symmetric functions, of which the first contains $\frac{m!}{\mu_0!\,\mu_1!\,\mu_2!\,..\,\mu_m!}$ terms, and the $(r+1)$st $\frac{m!}{(\mu_0-1)!\,\mu_1!\,\mu_2!..(\mu_r+1)!..\mu_m!}$ terms. These numbers can be written as $\frac{1}{\mu_0}\lambda$, and $\frac{1}{\mu_r+1}\lambda$, where $\lambda=\frac{m!}{(\mu_0-1)!\,\mu_1!\,\mu_2!..\mu_r!..\mu_m!}$.

But by this mode of summation, the first function will be repeated $\frac{n!}{\frac{1}{\mu_0}\lambda}=\mu_0\frac{n!}{\lambda}$ times, and the $(r+1)$st function $(\mu_r+1)\frac{n!}{\lambda}$ times.

We get therefore, by our summation, dividing by the common factor $n!/\lambda$,

$$\mu_0 b_0{}^m\Sigma(\beta\mu_0)^m(\beta\mu_1)^{m-1}(\beta\mu_2)^{m-2}\ldots.(\beta\mu_r)^{m-r}\ldots.(\beta\mu_m)^0+$$

$$(b_0)^{m-1}\sum_{r=1}^{r=n}b_r(1+\mu_r)\Sigma[\beta(\mu_0-1)]^m(\beta\mu_1)^{m-1}(\beta\mu_2)^{m-2}..[\beta(\mu_r+1)]^{m-r}..(\beta\mu_m)^0\equiv 0.$$

If in this equation we pick out the coefficient of $b_0{}^{\lambda_0}b_1{}^{\lambda_1}b_2{}^{\lambda_2}\ldots.b_n{}^{\lambda_n}$ we get,

$$\mu_0\begin{pmatrix}0^{\lambda_0}\,1^{\lambda_1}\,2^{\lambda_2}\ldots\ldots n^{\lambda_n}\\ 0^m m^{\mu_0}(m-1)^{\mu_1}..0^{\mu_m}\end{pmatrix}=-\sum_{r=1}^{r=n}(1+\mu_r)\begin{pmatrix}0^{\lambda_0}\,1^{\lambda_1}\,2^{\lambda_2}\ldots\ldots\ldots r^{\lambda_r-1}\ldots\ldots\ldots n^{\lambda_n}\\ 0^{m-1}m^{\mu_0-1}(m-1)^{\mu_1}..(m-r)^{\mu_r+1}..0^{\mu_m}\end{pmatrix}$$

$$=-\sum_{r=1}^{r=n}(1+\mu_r)\begin{pmatrix}0^{\lambda_0+1}\,1^{\lambda_1}\,2^{\lambda_2}\ldots\ldots r^{\lambda_r-1}\ldots\ldots n^{\lambda_n}\\ 0^m m^{\mu_0-1}(m-1)^{\mu_1}..(m-r)^{\mu_r+1}..0^{\mu_m}\end{pmatrix}.$$

3. Correspondence of the Formula with the Recurrence Formula in the Resultant Theory.

The preceding formula of 2 corresponds to the recurrence formula of A, 4, in the resultant theory. The identity of the two can be shown by eliminating the coefficients of the resultant from the latter formula by means of the relation [A, 5, (2)]

$$(m-\varkappa_1)(m-\varkappa_2)\ldots.(m-\varkappa_n) \mid 0^{\lambda_0}\, 1^{\lambda_1}\, 2^{\lambda_2}\ldots.n^{\lambda_n}$$

$$=(-1)^{\varkappa_1+\varkappa_2+\ldots\ldots+\varkappa_n}\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} & \ldots\ldots & n^{\lambda_n}\\ 0^m & \varkappa_1 & \varkappa_2\ldots. & \varkappa_n\end{pmatrix}$$

after suitably changing the notation, or we may first write the formula of 2 as

$$c_0\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} & \ldots\ldots & n^{\lambda_n}\\ 0^m & \varkappa_1 & \varkappa_2\ldots. & \varkappa_n\end{pmatrix}=-\sum_{r=1}^{r=n}(1+c_r)\begin{pmatrix}0^{\lambda_0+1} & 1^{\lambda_1} & \ldots\ldots & n^{\lambda_n}\\ 0^m & (\varkappa_1-r) & \varkappa_2\ldots. & \varkappa_n\end{pmatrix},$$

where at least $\varkappa_1=m$, $\varkappa_n=0$, and c_r is the number of β's which have the exponent $m-r$.

4. The Case where $n>m$.

In case $n>m$ we may use one of the relations

$$\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} & \ldots\ldots & n^{\lambda_n}\\ 0^m & \varkappa_1 & \varkappa_2\ldots. & \varkappa_n\end{pmatrix}=\begin{pmatrix}\varkappa_1 & \varkappa_2 & \ldots\ldots & \varkappa_n\\ 0^n & 0^{\lambda_0} & 1^{\lambda_1}\ldots. & n^{\lambda_n}\end{pmatrix}$$

$$=(-1)^{mn}\begin{pmatrix}(m-\varkappa_1)(m-\varkappa_2)\ldots\ldots(m-\varkappa_n)\\ 0^n 0^{\lambda_n} 1^{\lambda_{n-1}}\ldots\ldots\quad\ldots\ldots\quad\ldots.n^0\end{pmatrix}$$

of B, 1, (2), and then compute as before by the reduction formula of 2.

[To be Concluded.]

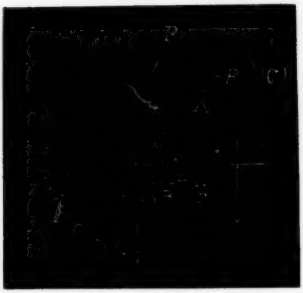
NOTE ON THE LOXODROMIC LINES OF THE TORUS.*

By Dr. ARNOLD EMCH, Professor of Graphic Mathematics, Kansas State Agricultural College, Manhattan, Kan.

1. Designating by u and v the angles, which, in Fig. 1, determine the position of a point B on the surface of a torus, the square of a linear element on the surface has the form,

$$ds^2=(R+r\sin v)^2(du_1^2+dv_1^2) \quad \ldots\ldots(1),$$

$$\text{where } u_1=u \text{ and } v_1=\int_0^v \frac{dv}{\frac{R}{r}+\sin v} \quad \ldots\ldots(2).$$



(C)=Revolved Meridian.

Fig. 1.

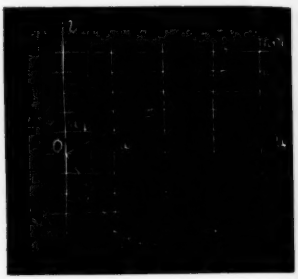


Fig. 2.

R and r are respectively the radii of the axial circle and of a meridian of the torus. As it is well known, by means of the expressions (2), the points of the torus are conformally transformed into the points of a rectangle. Considering exclusively the case where $R>r$ the integral v_1 has the value

$$v_1=\frac{r}{\sqrt{R^2-r^2}}\arcsin\left(\frac{1+\frac{R}{r}\sin v}{\frac{R}{r}+\sin v}\right) \quad \ldots\ldots(3).$$

The values of u and v can each vary from 0 to 2π. In this interval u_1 varies from 0 to 2π and according to the expression (3), v_1 from

$$\frac{r}{\sqrt{R^2-r^2}}\arcsin\frac{r}{R} \text{ to } \frac{r}{\sqrt{R^2-r^2}}\left[\arcsin\frac{R}{r}+2\pi\right].$$

*A preliminary statement concerning orthographic loxodromics on a torus was made by the author in a paper, "Ueber orthogonale Systeme und einige technische Anwendungen," which appeared in the program of the Polytechnic of Biel, 1898.

The sides of the primitive rectangle are therefore

$$w=2\pi \text{ and } w_1=\frac{r2\pi}{\sqrt{R^2-r^2}} \ldots\ldots(4).$$

To the lines $u_1=$const., $v_1=$const., respectively, parallel to the sides of the rectangle (w, w_1), correspond on the torus the meridians and parallels, and vice versa to the meridians and parallels $u=$const. and $v=$const., correspond in the rectangle lines, respectively, parallel to the sides w and w_1.

2. Evidently these lines form isothermal systems. If we now put $u_1+iv_1=z$, and if α and β designate two complex quantities $\alpha=p-iq$, $\beta=r(1+i)$, the monogenic function

$$\varphi+i\psi=\alpha z+\beta$$

represents also an isothermal system in the (u_1, v_1)-plane. There is

$$\varphi=au_1+bv_1+c \ldots\ldots(5),$$
$$\psi=-bu_1+av_1+c \ldots\ldots(6),$$

which represent two perpendicular pencils of parallel rays. Interpreted by means of formulae (2) they represent two systems of orthographic loxodromic lines on the torus.

3. Among the loxodromic lines of the torus we shall consider those that close after a certain number of revolutions around the axis and the axial circle of the torus. For this purpose it will be well to refer to the integral

$$Z=\int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \ldots\ldots(7),$$

by which the positive part of the z-plane is transformed into the rectangle of the z-plane whose sides correspond to the periods of the elliptic function

$$z=\lambda(Z) \ldots\ldots(8),$$

as defined by the elliptic integral (7).*

In order that the sides of the rectangle assume the values w and w_1, the modulus k of integral (7) must be chosen in such a manner that according to a well-known formula

$$k=4\sqrt{q}\left[\frac{(1+q^2)(1+q^4)(1+q^6)\ldots.}{(1+q)(1+q^3)(1+q^5)\ldots.}\right]^4,$$

where $q=e^{-(2\pi w_1)/w}$.

*See F. Klein, Ueber Riemann's Theorie der Algebraische Funktionen und ihrer Integrale, pages 50–55.

Supposing that $z=\lambda(Z)$ has been determined in this way, it is possible to transform the surface of the torus conformally upon the positive part of the z-plane and to express the periodic character of this transformation. The general period of $z=\lambda(Z)$ has the form

$$W=mw+niw,$$

or also

$$W=2\pi\left(m+ni\,\frac{r}{\sqrt{R^2-r^2}}\right).....(9).$$

By means of this expression for the period it is easy to show an interesting relation which exists between two orthographic loxodromic lines of a torus. From the origin O of the system of parallelograms of periods draw two lines, one to the point $\left(2m\pi,\ 2ni\pi\frac{r}{\sqrt{R^2-r^2}}\right)$; other to point $\left(2m_1\pi,\ -2n_1i\pi\frac{r}{\sqrt{R^2-r^2}}\right)$ of the system, (Fig. 2). The trigonometric tangents of these lines with the positive part of the u_1-axis are

$$u=\frac{n}{m}\cdot\frac{r}{\sqrt{R^2-r^2}},\quad v=-\frac{n_1}{m_1}\cdot\frac{r}{\sqrt{R^2-r^2}},$$

and the tangent of the angle included by the two lines is

$$\frac{\frac{n}{m}\cdot\frac{r}{\sqrt{R^2-r^2}}+\frac{n_1}{m_1}\cdot\frac{r}{\sqrt{R^2-r^2}}}{1-\frac{n}{m}\cdot\frac{n_1}{m_1}\cdot\frac{r}{\sqrt{R^2-r^2}}}.$$

The condition for a right angle is

$$\frac{n}{m}\cdot\frac{n_1}{m_1}=\frac{r}{\sqrt{R^2-r^2}}.....(10).$$

In this case the corresponding loxodromic lines on the torus are orthographic and since in the plane (u_1, v_1) the lines end in points of periods, it is clear that the two orthographic loxodromic lines close. If the numbers m and n are given, and provided R/r is a rational fraction, then it is always possible to find two integers m_1 and n_1 so that relation (10) is satisfied. Evidently m gives the number of revolutions around the axis (perpendicular to the parallels) and n the number of revolutions around the axial circle of the torus. Thus we are led to the theorem :

If the ratio R/r of a torus is a rational fraction and if a loxodromic line of the torus closes after m revolutions around the axis and n revolutions around the

axial circle of the torus, then every orthographic loxodromic line closes after m_1 and n_1 revolutions around the same axes, so that

$$\frac{n}{m} \cdot \frac{n_1}{m_1} = \frac{r}{\sqrt{R^2 - r^2}}.$$

4. Among the great number of special cases, which arise from this relation, we shall simply mention the case where $m=n=1$. The following theorem is easily found:

The loxodromic lines of a torus, which turn once around the axis and once around the axial circle of the torus, are the circles formed by the intersection of the double-tangent planes with the torus.

The results found about loxodromic lines of a torus may immediately be generalized for those cyclides which arise from the torus by circular transformations.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

108. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A man who feels his death approach bequeaths to his young wife one-third of his fortune, and the remaining two-thirds to his son, if such should be born; but one-half of it to the widow and the other half to his daughter, if such should be born. After his death, twins are born, a son and a daughter. How should the fortune be divided amongst the three?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.; P. S. BERG, B. S., Principal of Schools, Larimore, N. D.; SYLVESTER ROBINS, North Branch, N. J.; JOSIAH H. DRUMMOND, LL. D., Portland, Me.; ELMER SCHUYLER, High Bridge, N. J.; and ALOIS F. KOVARIK, Professor of Mathematics and Physics, Decorah Institute, Decorah, Ia.

By the conditions of the problem, the wife is to have as much as the daughter and half as much as the son.

$\therefore$ son : wife : daughter $= 2 : 1 : 1$.

$\therefore$ son should receive $\frac{1}{2}$ the fortune, wife should receive $\frac{1}{4}$ the fortune, and daughter should receive $\frac{1}{4}$ the fortune.

II. Solution by D. G. DORRANCE, Camden, N. Y.

Consider the fortune (a) divided into two fortunes each of one-half the value of the whole, viz: $\frac{1}{2}a$ and $\frac{1}{2}a$.

Then widow's share of first $\frac{1}{3}a$ would be $\frac{1}{3}$ of $\frac{1}{2}a=\frac{1}{6}a$, and the son's share would be $\frac{2}{3}$ of $\frac{1}{2}a=2a/6$ or $\frac{1}{3}a$.

Widow's share of second $\frac{1}{4}a$ would be $\frac{1}{2}$ of $\frac{1}{2}a=\frac{1}{4}a$, and the daughter's share would be $\frac{1}{2}$ of $\frac{1}{4}a=\frac{1}{8}a$.

$\therefore$ Widow should receive $\frac{1}{6}a+\frac{1}{8}a=5a/12$. Son should receive $4a/12$. Daughter should receive $3a/12$.

[See Vol. I., page 127, for solution of a similar problem. Also see Cajori's *History of Mathematics*, page 79. ED. F.]

109. Proposed by B. F. FINKEL, A. M., M. S., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Why do fences and telegraph poles appear to move rapidly in an opposite direction to one traveling in a railway car? [From Moore's *Grammar School Arithmetic*, page 150.]

I. Answer by the PROPOSER.

When we are at rest and observing a moving object, a line from our eye to the object projects the object in different points on the landscape, or, if the object is in the sky, as, for example, a cloud, the line from our eye to the cloud projects the cloud in different points of the sky. By noticing the different projected positions of the object, we become conscious of its motion. Now if we are moving and are looking at some stationary object, the line from our eye to the object again projects the object in different points of the landscape, the appearance being the same as in the case where the object moved and we were stationary. But in this case the apparent motion of the object is opposite to our real motion.

When we are unconscious of our own motion, as, for example, when riding in a car over a smooth road, we attribute the appearance of an object projected in different positions on the landscape to the motion of the object, and since our motion is direct, the apparent motion of the object is retrograde. This motion is more striking as the object upon which our attention is fixed is the nearer, for the reason that the angle of parallax is greater.

II. Answer by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

When we are standing on the street watching a moving object, the line from the eye to the object is constantly changing its direction causing us to turn our head in order to follow the object with our eye. When we are sitting in a railway train, the train is moving with us in it and seems to us as if it were still and the objects passing the same as when we were standing on the street. In either case we must turn the head, hence we seem to see the objects passing us instead of our passing the objects.

III. Answer by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

On the train we are stationary with respect to the cars, but in motion with respect to the earth. Noticing the things next to us not to be in motion, while

the posts, etc., outside the car seem passing by on account that we judge mostly by what is near our eyes. It is this that being stationary with respect to the earth, the ancients imagined the sun and stars to move around the earth instead of the opposite.

GEOMETRY.

110. Proposed by P. S. BERG, A. M., Principal of Schools, Larimore, N. D.

If the three face angles of the vertical triedral angle of a tetraedron are right angles, and the lengths of the lateral edges are represented by a, b, and c, and of the altitude by p, then $1/p^2=1/a^2+1/b^2+1/c^2$. [Chauvenet's *Geometry*.]

I. Solution by GEORGE R. DEAN, Professor of Mathematics, School of Mines and Metallurgy, University of Missouri, Rolla, Mo.; P. H. PHILBRICK, C. E., Lake Charles, La.; and CHAS. C. CROSS, Libertytown, Md.

Let OC, OB, OA be the edges mutually at right angles; OP the perpendicular. Join AP and produce to meet CB at D; join BP and produce to meet AC at E; join CP and produce to meet AB at F.

Then $$\frac{\triangle APC}{\triangle ABC}+\frac{\triangle BPC}{\triangle ABC}+\frac{\triangle APB}{\triangle ABC}=1.$$

But $$\frac{\triangle PAC}{\triangle ABC}=\frac{PE}{BE},\quad \frac{\triangle BPC}{\triangle ABC}=\frac{DP}{AD},$$

$$\frac{\triangle BPA}{\triangle ABC}=\frac{PF}{CF}.$$

Therefore $$\frac{PE}{BE}+\frac{DP}{AD}+\frac{PF}{CF}=1.$$

$$BE\times BP=OB^2=b^2;\quad BE=\frac{b^2}{BP};\quad \frac{PE}{BE}=\frac{PE\times BP}{b^2}=\frac{OP^2}{b^2}.$$

$$AD\times AP=OA^2=a^2;\quad AD=\frac{a^2}{AP};\quad \frac{DB}{AD}=\frac{AP\times DP}{a^2}=\frac{OP^2}{a^2}.$$

$$CF\times CP=OC^2=c^2;\quad CF=\frac{c^2}{CP};\quad \frac{PF}{CF}=\frac{PF\times PC}{c^2}=\frac{OP^2}{c^2}.$$

Hence $$\frac{OP^2}{a^2}+\frac{OP^2}{b^2}+\frac{OP^2}{c^2}=1,\text{ or }\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$ Q. E. D.

II. Solution by P. S. BERG, B.Sc., Superintendent of Schools, Larimore, N. D.; WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; ELMER SCHUYLER, High Bridge, N. J.; W. H. WILSON, Professor of Mathematics, Geneva College, Beaver Falls, Pa.; and G. B. M. ZERR, A. M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $OP=p$, $OC=a$, $OA=b$, $OB=c$, $OI=l$, $IH=m$, $PH=n$.

The triangles (right angled) OPA and OPI are similar, having $\angle POA$ in common.

$\therefore p/b=l/p$, or $p^2/b^2=l^2/p^2$, or $p^4/b^2=l^2$.

Similarly, $p^4/a^2=n^2$, $p^4/c^2=m^2$.

$\therefore p^4/a^2+p^4/b^2+p^4/c^2=n^2+l^2+m^2=p^2$.

$\therefore 1/a^2+1/b^2+1/c^2=1/p^2$.

Also solved by *J. SCHEFFER*, and *E. D. SCALES*.

111. Proposed by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Given that the area of a triangle is equal to half the product of two sides and the sine of the included angle, prove that $\sin(x+y)=\sin x\cos y+\cos x\sin y$.

I. Solution by W. F. BRADBURY, A. M., Head Master, Cambridge Latin School, Cambridge, Mass., and B. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, O.

Given area $\triangle=\frac{1}{2}ac\sin B$.... .(1).

$\sin B=\sin[180^\circ-(A+C)]=\sin(A+C)$.

$\therefore$ Area $\triangle=\frac{1}{2}ac[\sin(A+C)]$.....(2).

Draw BD perpendicular to AC.

Area $\triangle=\frac{1}{2}BD(AD+DC)$.

But $BD=c\sin A=a\sin C$, and $AD=c\cos A$, and $DC=a\cos C$.

$\therefore$ Area $\triangle=\frac{1}{2}\begin{matrix}c\sin A\\ \text{or}\\ a\sin C\end{matrix}(c\cos A+a\cos C)=\frac{1}{2}(ac\sin C\cos A+ac\sin A\cos C)\ldots(3)$.

Putting (2)=(3), $\frac{1}{2}ac[\sin(A+C)]=\frac{1}{2}ac(\sin A\cos C+\cos A\sin C)$ or $\sin(A+C)=\sin A\cos C+\cos A\sin C$.

II. Solution by J. OWEN MAHONEY, B. E., Professor of Mathematics and Science, Carthage High School, Carthage, Tex.; J. W. YOUNG, Columbus, O.; J. SCHEFFER, A. M., Hagerstown, Md.; E. L. SHERWOOD, A. M., Professor of Mathematics, Whiteworth College, Miss.; WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; B. F. SINE, Principal of Normal School, Capon Bridge, W. Va.; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; JOHN MACHNIE, A. M., Professor of Latin, University of North Dakota; H. F. STRATTON, Student in Heidelberg University, Tiffin, O.; J. C. NAGLE, C. E., Professor of Civil Engineering and Physics in the Agricultural and Mechanical College of Texas, College Station, Tex.; CHARLES C. CROSS, Libertytown, Md.; and ELMER SCHUYLER, High Bridge, N. J.

PROOF. Consider the triangle ACB.

The area of $ACB=\frac{1}{2}ac\sin B=\frac{1}{2}ac\sin(A+C)=\frac{1}{2}hb=\frac{1}{2}h(AD+DC)$, or

$$\sin(A+C)=\frac{h}{a}\cdot\frac{AD}{c}+\frac{h}{c}\cdot\frac{DC}{a}$$

$$=\sin C\cos A+\sin A\cos C.$$

$\therefore \sin(x+y)=\sin x\cos y+\cos x\sin y$.

III. Solution by the PROPOSER.

Let ABC and CBD be the two angles x and y. Draw any line perpendicular to BC meeting AB, BC, BD at P, Q, R.

Then $\triangle BPR = \triangle BPQ + \triangle BQR$.

$2\triangle BPR = BP \times BR \times \sin PBR = BP.BR.\sin(x+y)$.

$2\triangle BPR = BP \times BQ \times \sin PBQ = BP.BQ.\sin x$.

$2\triangle BQR = BQ.RB.\sin y$.

Hence $BP.BR\sin(x+y) = BP.BQ\sin x + BQ.BR\sin y$.

Dividing by $BP.BR$, $\sin(x+y) = \frac{BQ}{BR}\sin x + \frac{BQ}{BP}\sin y = \cos y \sin x + \cos x \sin y$.

CALCULUS.

84. Proposed by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Find the equation of the curve upon which a given ellipse must roll in order that one of its foci may describe a straight line.

Solution by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Let θ be the angle between the major axis of the ellipse and the radius vector from the focus to the point of contact; r the length of this radius, a, b, semi-axes, e eccentricity. Then

$$r = \frac{a(1-e^2)}{1-e\cos\theta}.$$

Let the axis of x be taken parallel to the given line. Then since the point of contact is the instantaneous center, the radius vector will always be perpendicular to the axis of x, and hence $r+y = a(1-e)$, y being the ordinate of a point on the required curve. We have also

$$r\frac{d\theta}{dr} = -\frac{dx}{dy}.$$

Eliminating r and θ, we find

$$y - \sqrt{b^2\left(\frac{dy}{dx}\right)^2 + a^2} = ae.$$

Whence $\frac{ae-y}{b} = \sin\left(\frac{x}{b} + \sin^{-1}\frac{ae}{b}\right)$, or $y' = b\sin\left(\frac{x'}{b}\right)$.

86. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.

Prove that the curve whose normal equals its radius of curvature drawn in an opposite direction, is the catenary $y = c\cosh(x/c)$.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

We should have

$$\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=y\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}},\text{ whence }\frac{2\frac{dy}{dx}\frac{dy^2}{dx^2}}{1+\frac{dy^2}{dx^2}}=\frac{2}{y}\frac{dy}{dx}.$$

Multiplying by dx and integrating and correcting,

$$1+\frac{dy^2}{dx^2}=\frac{y^2}{c^2}.$$

This gives $dx=\frac{cdy}{\sqrt{y^2-c^2}}$; whence integrating and correcting again,

$$x=c\log\left(\frac{y+\sqrt{(y^2-c^2)}}{c}\right),\text{ or } y=c\cosh(x/c).$$

II. Solution by MARY M. BLAINE, B. Sc., Graduate Student, Drury College, Springfield, Mo.

Our differential equation, formed by substituting the values of normal and ρ, the radius of curvature is

$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\ldots\ldots(1).$$

$$\therefore y\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]\ldots\ldots(2).$$

Now let $p=\frac{dy}{dx}$, then $\frac{d^2y}{dx^2}=\frac{dp}{dx}=\frac{dp}{dy}\cdot\frac{dy}{dx}=\frac{dp}{dy}\cdot p$.

Substituting, $y\,p\,\frac{dp}{dy}=(1+p^2)\ldots\ldots(3)$.

$$\therefore\frac{pdp}{1+p^2}=\frac{dy}{y}.\quad\text{Integrating,}$$

$\log y=\frac{1}{2}\log(p^2+1)+\log c_1$, or $\log(y/c_1)=\log\sqrt{(p^2+1)}\ldots\ldots(4)$.
$\therefore y/c_1=\sqrt{(p^2+1)}\ldots\ldots(5)$.
Squaring and clearing, $c_1p^2=y^2-c_1^2\ldots\ldots(6)$.

$$p^2=\frac{y^2-c_1^2}{c_1^2},\text{ or }p=\frac{\sqrt{(y^2-c_1^2)}}{c_1^2}\ldots\ldots(7).$$

Then, substituting for p, and separating,

$$\frac{dy}{\sqrt{y^2-c_1^2}}=\frac{dx}{c_1^2}\ldots\ldots(8).\quad\text{Integrating, }\log[y+\sqrt{(y^2-c_1^2)}]=\frac{x}{c_1^2}+\log c_2\ \ldots\ldots(9).$$

$$\therefore\frac{y+\sqrt{(y^2-c_1^2)}}{c_2}=e^{x/c_1}\ldots\ldots(10).$$

Transposing and squaring, $y^2-c_1^2=c_2^2e^{2x/c_1}-c_2e^{x/c_1}y+y^2\ldots\ldots(11).$

$$\therefore\ y=\frac{c_2^2e^{2x/c_1}+c_1^2}{c_2 2e^{x/c_1}}\ldots\ldots(12),\text{ or }y=\frac{c_2^2e^{x/c_1}+c_1^2e^{-(x/c_1)}}{2c_2}\ldots\ldots(13).$$

Let $c_2=c_1$, thus moving the origin to the right. Then $y=\frac{c_1[e^{x/c_1}+e^{-(x/c_1)}]}{2}$

$\ldots\ldots(14)$, or $y=c_1\cosh(x/c_1)$, which is the equation of the catenary.

Also solved by *J. W. YOUNG*.

87. Proposed by **MARY M. BLAINE**, B. Sc., Graduate Student, Drury College, Springfield, Mo.

Integrate $(px-y)(py+x)=h^2p$, where $p=dy/dx$.

I. Solution by **WILLIAM HOOVER**, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

We have $(px-y)(py+x)=h^2p\ldots\ldots(1)$, or

$$p^2xy-py^2+px^2-xy=h^2p\ldots\ldots(2).$$

Multiplying by y and arranging,

$$y^2=\left(y\frac{dy}{dx}\right)^2-\left(\frac{y^3}{x^2}-xy+\frac{h^2y}{x}\right)\frac{dy}{dx}\ldots\ldots(3).$$

Putting $y^2=y'$, $x^2=x'$,

$$y'=x'\left(\frac{dy'}{dx'}\right)^2-y'\frac{dy'}{dx'}+x'\frac{dy'}{dx'}-\frac{dy'}{dx'}h^2\ldots\ldots(4),$$

or $y'=p'x'-\frac{h^2p'}{p'+1}\ldots\ldots(5)$, Clairaut's Form, giving $y^2-cx^2=-\frac{ch^2}{c+1}\ \ldots\ldots(6).$

II. Solution by **GEORGE R. DEAN**, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Divide by p, differentiate and reduce to

$$xy\frac{dp}{dx}+(px-y)p=0.$$

This may be written

$$\frac{\frac{d}{dx}(px-y)}{px-y}=-\frac{1}{y}\frac{dy}{dx},$$

the integral of which is $pxy-y^2=c$.

Eliminating p between this and the given equation,

$$y^2+\frac{c}{c-h^2}=-c.$$

Putting $\frac{c}{c-h^2}=-c'$, $y^2-c'x^2=\frac{c'h^2}{1+c'}$.

This solution may also be obtained by solving $pxy-y^2=c$, obtaining $y^2+c'x^2=c$.

This gives $p=-\frac{c'x}{y}$, which substituted in the given equation makes $c=\frac{-c'h^2}{1+c'}$.

Also solved by the *PROPOSER*.

MECHANICS.

79. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

The four wheels of a street car are rigidly fixed to their axles so that axles and wheels turn together. Is it more advantageous to apply the brakes to the front or to the rear wheels, supposing the brakes to block the wheels in each case?

Solution by the PROPOSER.

The question may be answered by treating the problem as a statical one, thus: Suppose the car placed upon an inclined plane and let us inquire in which case may the plane be raised to the greater angle before slipping begins. Let $2a$ be the distance between the centers of the wheels, each of radius c, b the distance of the center of gravity, G of the car above (or below) this line of centers, w the weight of each of the trucks, w_1 the weight of car.

Take the case first when brakes are applied to rear wheels, there being in this case, of course, no friction between the front wheels and plane. Consider the figure as consisting of two rigid bodies, viz, the front trucks, and the car with the rear trucks. The forces acting upon the front trucks are their weight w, the reaction R of the plane, and a force, P, at O obliquely downward, which is the resultant of w and the backward pull of the car and rear wheels. Upon the car at this point O there will also be an equal and opposite force to P, the resultant of R and w.

The forces on car and rear wheels are, this force P at O, w_1 at G, w at O', R_1 the reaction of the plane, and F the force of friction.

From forces on front trucks we have,

(1). $w\sin\theta = P\cos\psi$
(2). $w\cos\theta = R - P\sin\psi$
} ψ being the angle between P and the line of centers OO'.

From forces on car and rear wheels, letting μ be coefficient of friction, taking moments about O' and resolving horizontally and vertically,

(3). $2aP\sin\psi = w_1(a+b\tan\theta)\cos\theta - \mu R_1 c.$

(4). $\mu R_1 = (w+w_1)\sin\theta + P\cos\psi.$

(5). $R_1 + P\sin\psi = (w+w_1)\cos\theta.$

From these five equations we get,

$$\tan\theta = \frac{a\mu(2w+w_1)}{aw_1 b + (2a-c\mu)(2w+w_1)} \ldots\ldots (A).$$

Next take the case when brakes are applied to front wheels, considering in this case, the rear trucks as one rigid body, and the car with front trucks as the other. The two oblique forces P now act at O' one upon the car and one upon rear wheels (lettered P' to avoid confusion).

From forces on rear wheels we get,

(a). $w\sin\theta = P'\cos\psi'$
(b). $w\cos\theta = R_1 - P'\sin\psi'$
} ψ' being the angle between P' and OO'.

From forces on car and front wheels, taking moments about O, resolving horizontally and vertically.

(c). $2aP'\sin\psi' = w_1(a-b\tan\theta)\cos\theta - \mu Rc.$

(d). $\mu R = (w+w_1)\sin\theta + P'\cos\psi'.$

(e). $R + P'\sin\psi' = (w+w_1)\cos\theta.$

From these five equations we get

$$\tan\theta = \frac{a\mu(2w+w_1)}{(2a-\mu c)(2w+w_1) - \mu w_1 b} \ldots\ldots (B).$$

Comparing (B) with (A) we see that in the latter case $\tan\theta$, and hence θ, is the greater, and hence we infer it is more advantageous to apply the brakes to the front wheels.

80. Proposed by B. F. FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A circular board is placed on a smooth horizontal plane and a boy runs with uniform speed around on the board close to the edge. Find the motion of the center of the board.

I. Solution by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

The system is not acted upon by any external force. Hence the center of gravity is stationary. Let I be the moment of inertia of the board about its center, m the mass of the board, m' that of the boy, r the distance of center of system from center of board, ω the angular velocity of board. Then $I\omega^2+mr^2\omega^2=m'v^2$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $AC=a$ be the radius of board, A the starting point of the boy, m his mass, P the mass of the board, G the center of gravity of the system at starting.

Since there are no horizontal forces *external* to the system, by D'Alembert's principle, the center of gravity of the system must move downwards; but this is impossible.

$\therefore$ The equation of motion of the center of gravity is, $m\frac{d^2\bar{x}}{dt^2}=0$.

$\therefore \frac{d\bar{x}}{dt}=0$, since both boy and board start from rest.

$\therefore \bar{x}=$ a constant. $\therefore$ The center of gravity of the system remains unaltered throughout the motion.

Hence since the distance between the center of the board and boy remain constant, the distances of the center of gravity from the boy and the center of the board respectively will remain constant.

$\therefore$ C describes a circle with $GC=\frac{ma}{m+P}$ as radius.

$\therefore$ A describes a circle with $AG=\frac{Pa}{m+P}$ as radius.

$\therefore$ both center of board and boy describe circles. If $m=P$ these circles will be equal and have AC for diameter.

MISCELLANEOUS.

69. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, P. O., Sebastopol, Cal.

Find the locus of a point equi-distant from the circumferences of two fixed circles.

I. Solution by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Denote the two given circumferences by K_1 and K_2; take their centers at $(-\frac{1}{2}a, 0)$ and $(\frac{1}{2}a, 0)$; denote their radii by r_1 and r_2, r_1 not being less than r_2. Let (x, y) denote a point in the desired locus.

Then if (x, y) is inside K_1 and inside K_2,

$$r_1-\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{r_2-\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}\}=0.....(1).$$

Or if (x, y) is outside K_1 and outside K_2,

$$\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-r_1-\{\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}-r_2\}=0,$$

which is the same as (1).

Similarly, if (x, y) is inside K_1 and outside K_2, or if (x, y) is outside K_1 and inside K_2,

$$r_1-\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}-r_2\}=0\ldots\ldots(2).$$

The desired locus is represented by the equation obtained by taking the product of (1) and (2). I am not aware of any general criterion as to the nature and limiting points of curves (or pieces of curve) whose equations are irrational. If either variable in the equation is an explicit function of the other, the nature of the curve, as well as the limiting points, can usually be detected by inspection; but considerable difficulty may be experienced in finding the nature and limiting points of the curve represented by an irrational equation in which x and y are implicit functions of each other.

Although aware that the method is exceedingly unsatisfactory, I will first include the curves or pieces of curve represented by the equations congeneric to (1) and (2) and afterward exclude such pieces thus incorrectly dragged in by the process of rationalization.

The equations, whose product make (1) rational, are:

$$r_1-\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{r_2-\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}\}=0\ldots\ldots(1).$$

$$r_1-\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{r_2+\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}\}=A\ldots\ldots(3).$$

$$r_1+\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{r_2-\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}\}=B\ldots\ldots(4).$$

$$r_1+\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{r_2+\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}\}=C\ldots\ldots(5).$$

The equations, whose product make (2) rational, are

$$r_1-\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}-r_2\}=0\ldots\ldots(2).$$

$$r_1-\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}+\{\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}+r_2\}=D\ \ldots\ (6).$$

$$r_1+\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}-\{\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}-r_2\}=E\ldots\ldots(7).$$

$$r_1+\sqrt{[(x+\tfrac{1}{2}a)^2+y^2]}+\{\sqrt{[(x-\tfrac{1}{2}a)^2+y^2]}+r_2\}=F\ldots\ldots(8).$$

The product of (1), (3), (4), and (5) will represent all lines or parts of lines the points of which satisfy *any one of them with its right hand member put equal to zero*: for if the right hand member of any of them, say (3) was zero, and the right hand member of (1) was, say G, precisely *the same product would be*

obtained; a similar statement may be made with reference to the product of (2), (6), (7), and (8), although in the case of (8) no points would be obtained by making the right hand member equal to zero, since the sum of two positive distances cannot equal zero.

Denote the difference of the radii (r_1-r_2) by d, and the sum (r_1+r_2) by s. The product of the first four equations is

$$4(a^2-d^2)x^2-4d^2y^2=d^2(a^2-d^2).....(9).$$

And the product of the second four equations is

$$4(a^2-s^2)x^2-4s^2y^2=s^2(a^2-s^2).....(10).$$

It will be remembered that a is the distance between the centers of the two given circles, d the difference, and s the sum of their radii. In eliminating from (9) the points not satisfying (1), and from (10) the points not satisfying (2), I will consider a to vary.

I. $a<d<s$ or K_2 is wholly inside K_1; in this case both (9) and (10) are equations of ellipses; (9) is the locus of centers of circles tangent internally to K_1 and enclosing K_2; these points satisfy (3) put equal to zero. Equation (10) is the locus of centers of circles tangent internally to K_1 and externally to K_2; all such points satisfy (2) and hence (10) is the equation of the desired locus.

II. If $a=d$ and $a<s$, K_2 is inside K_1 and internally tangent to it; in this case, equation (9) reduces to $y=0$ or the axis of X. From $x=+\infty$ to $x=+\frac{1}{2}a$, the points satisfy (1); from $x=+\frac{1}{2}a$ to $x=-\frac{1}{2}a$, the points satisfy (3) put equal to zero; from $x=-\frac{1}{2}a$ to $x=-\infty$, the points satisfy (5) put equal to zero. Equation (10) as in Case I. The locus consists then of (9), from $x=+\infty$ to $x=+\frac{1}{2}d$, and of (10).

III. If $a>d$ and $a<s$, K_1 and K_2 intersect; in this case equation (9) represents a hyperbola, all the points on the right hand branch satisfying (1), and the points on the left hand branch satisfying (5) put equal to zero. Equation (10) as in Cases I and II. The locus consists of (9), $x=+\infty$ to $x=+\frac{1}{2}d$, and of (10).

IV. If $a<d$ and $a=s$, K_1 and K_2 are tangent externally. Equation (9) as in Case III. Equation (10) reduces to $y=0$, or the axis of X. From $x=+\infty$ to $x=+\frac{1}{2}a$, the points satisfy (6); from $x=+\frac{1}{2}a$ to $-\frac{1}{2}a$, the points satisfy (2); and from $x=-\frac{1}{2}a$ to $-\infty$, the points satisfy (7). The locus consists of (9) from $x=+\infty$ to $x=+\frac{1}{2}d$, and of (10) from $x=+\frac{1}{2}a$ to $x=-\frac{1}{2}a$.

V. If $a>d$ and $a>s$, K_1 and K_2 are exterior to each other. Equation (9) as in Cases III and IV. Equation (10) also represents a hyperbola, the points on the right hand branch satisfying (6) put equal to zero, and the points on the left hand branch satisfying (7) put equal to zero. The locus consists of (9) from $x=+\infty$ to $x=+\frac{1}{2}d$.

In many cases the limiting points may be found by equating the radical to zero if it is of the second degree as in this case. But this is not true as a gener-

al proposition. For the right hand branch of the hyperbola (9) in III, IV, and V crosses the circle at $x=+\frac{1}{2}a$ and is inside from $x=+\frac{1}{2}a$ to $x=+\frac{1}{2}d$. The left hand branch begins at $x=-\frac{1}{2}d$ and leaves the circle at $x=-\frac{1}{2}a$. Between the limits suggested by equating the radicals to zero ($x=+\frac{1}{2}a$ and $x=-\frac{1}{2}a$) there are two pieces of curve, one of which contains points filling the given conditions, the other containing points violating them. That the tentative methods adopted by me are wholly unsatisfactory I freely admit, and yet I feel sure that graphical methods will be of great aid in establishing a theory of irrational equations.

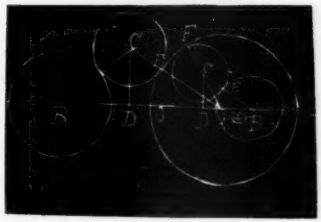
II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $AF=R$, $BE=r$, $CF=CE=t$.

Then when B is inside of circle A we have $BC+AC=r+t+R-t=R+r$.

$\therefore$ The locus of C, the point equidistant from both circumferences, is an ellipse, foci A, B.

When B is without A we have $AC-BC=R+t-r-t=R-r$.



$\therefore$ The locus of C is a hyperbola, foci A, B. Also as follows:

Let the mid-point O, between A and B be origin, $AB=2a$, $OD=x$, $CD=y$.

$$\therefore\ BC^2=(r+t)^2=(a\pm x)^2+y^2\ldots.(1).$$

$$AC^2=(R\mp t)^2=(x\mp a)^2+y^2\ldots(2).$$

Eliminating t between (1) and (2) we get

$$r\pm R\mp 1\sqrt{[(x\mp a)^2+y^2]}=1\sqrt{[(x\pm a)^2+y^2]},\text{ or }\frac{4x^2}{(R\pm r)^2}+\frac{4y^2}{(R\pm r)^2-4a^2}=1.$$

This equation shows that when one circle is within the other $(R+r)>2a$ and the locus is an ellipse with semi-axes $\frac{1}{2}(R+r)$ and $\frac{1}{2}1\sqrt{[(R+r)^2-4a^2]}$.

When circles are external the locus is an hyperbola with semi-axes $\frac{1}{2}(R-r)$ and $\frac{1}{2}1\sqrt{[4a^2-(R-r)^2]}$.

If the circles intersect, for internal contact to one circle and external to the other we have the same ellipse; for both internal and both external we have the same hyperbola as above.

Also solved by *GUY B. COLLIER, CHARLES C. CROSS, WALTER H. DRANE, ALOIS F. KOVARIK, W. W. LANDIS, J. SHEFFER*, and *COOPER D. SCHMITT*.

DIOPHANTINE ANALYSIS.

73. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find integral values for x and y in $\begin{pmatrix}2x^2-y^2=\square\\ 2y^2-x^2=\square\end{pmatrix}$.

Discussion by W. T. KING, Ottawa, Canada.

We may suppose x and y to have no common factor ; for if they have one it will also be a common factor of a and b, and its square may be divided out of both equations, leaving equations of the same form. For the same reason no two of x, y, a, or of x, y, b can have a common factor, nor can a and b, without its being common to all. We therefore suppose that x, y, a and b are all prime to one another. Also we may assume without loss of generality that $x>y$. Then it is evident that $a>x$, $b<y$.

From the first equation we have

$$2(a^2-x^2)=a^2-y^2.$$

Suppose $\frac{2(a-x)}{a-y}$ when reduced to its lowest terms to be equal to p/q, so that p and q are prime to one another.

Then $2q(a-x)=p(a-y)$ and $p(a+x)=q(a+y)$.

Whence $(p^2+2q^2)a+(p^2-2q^2)x=2pqa$ or $x/a=\frac{p^2+2q^2-2pq}{2q^2-p^2}$.

The numerator being evidently positive, the denominator $(2q^2-p^2)$ must be also.

Since x/a is in its lowest terms, the fraction $\frac{p^2+2q^2-2pq}{2q^2-p^2}$, when reduced to its lowest terms, must have its numerator equal to x, and its denominator to a. But it is evident that p and q being prime to one another, the numerator and denominator can have no common factor, unless it be 2, in the event of p being even. Therefore if p be odd,

$$x=p^2+2q^2-2pq \ldots\ldots (1).$$

$$y=2q^2-p^2 \ldots\ldots (2).$$

Whence by the equation $q(a+y)=p(a+x)$ we get

$$y=4pq-2q^2-p^2 \ \ldots\ (3).$$

If p be even,

$$x=\tfrac{1}{2}(p^2+2q^2-2pq)=2(\tfrac{1}{2}p)^2+q^2-2(\tfrac{1}{2}p)q,$$
$$a=\tfrac{1}{2}(2q^2-p^2)=q^2-2(\tfrac{1}{2}p)^2,$$
$$y=\tfrac{1}{2}(4pq-2q^2-p^2)=4(\tfrac{1}{2}p)q-q^2-2(\tfrac{1}{2}p)^2.$$

These equations are of the same form as the preceding, except in the case of a. For x and y we may take them in all cases,

$$\left.\begin{aligned} x&=p^2+2q^2-2pq \\ y&=4pq-2q^2-p^2 \end{aligned}\right\} \ldots\ldots (4).$$

Now from the equation $2y^2-x^2=b^2$, we have

$$2(y^2-b^2)=x^2-b^2.$$

Whence as before we have,

$$2s(y-b)=r(x-b), \quad r(y+b)=s(x+b),$$

r and s being prime to one another. Whence as before,

$$\left.\begin{array}{l} x=4rs-2s^2-r^2 \\ y=2s^2+r^2-2rs \\ b=2s^2-r^2, \text{ or } =s^2-2(\tfrac{1}{2}r)^2, \text{ when } r \text{ is even,} \end{array}\right\} \text{or quantities of the same form} \quad \ldots\ldots(5).$$

Equating the two values of x and y we have $2pq=2rs$.

$\therefore x-pq=x-rs$ or $p^2+2q^2-3pq=3rs-2s^2-r^2$ or $(2q-p)(q-p)=(2s-r)(r-s)$.

Now from the equation $p(a+x)=q(a+y)$ we see that $q>p$, for $x>y$.
Then $q-p$ and $2q-p$ are both positive.
Similarly from the equation $r(y+b)=s(x+b)$, we see that $r>s$.
And since $(2s-r)(r-s)$ is positive, $2s-r$ is also positive.
$\therefore r>s<2s$. Now we may write

$$\left.\begin{array}{l} n(q-p)=m(r-s) \\ m(2q-p)=n(2s-r) \end{array}\right\} \ldots\ldots(6).$$

where m and n are positive integers, prime to one another.

$$\therefore (2m^2+n^2)q-(m^2+n^2)p=mns,$$

$$2(m^2+n^2)q-(m^2+2n^2)p=mnr.$$

$\therefore [(2m^2+n^2)q-(m^2+n^2)p][2(m^2+n^2)q-(m^2+2n^2)p]=m^2n^2rs=m^2n^2pq$, or

$$2(m^2+n^2)(2m^2+n^2)q^2-2(2m^2+n^2)(m^2+2n^2)pq+(m^2+n^2)(2n^2+m^2)p^2=0.$$

Since the value of q/p derived from this equation must be rational,

$$(2m^2+n^2)^2(m^2+2n^2)-2(m^2+n^2)(2m^2+n^2)(m^2+2n^2)$$

must be a square, *i. e.* $(2m^2+n^2)(m^2+2n^2)m^2n^2$ must be a square, and $(2m^2+n^2)(m^2+2n^2)$ must be a square.

Therefore either $2m^2+n^2$ and m^2+2n^2 are both squares, or when divided by a common factor, the quotients are squares.

In the first case let $2m^2+n^2=h^2$, $m^2+2n^2=k^2$, then $m^2+n^2=\frac{1}{3}(h^2+k^2)$.

Now the sum of two squares is not divisible by 3, unless each square is divisible by 3, and therefore by 9. Hence $\frac{1}{3}(h^2+k^2)$ is an integer, whence $\frac{1}{3}(m^2+n^2)$ is an integer ; whence m and n are both divisible by 3, which is con-

trary to the hypothesis that m and n have no common factor. Therefore $2m^2+n^2$ and m^2+2n^2 have a common factor. This factor must be a factor also of their sum, *i. e.* $3(m^2+n^2)$.

But m^2+n^2 is prime to $2m^2+n^2$, since if they have a common factor it is also a factor of m^2+n^2 and $(2m^2+n^2)-(m^2+n^2)$, or of m^2+n^2 and m^2, or of n^2 and m^2, which is again contrary to hypothesis.

Therefore the common factor of $2m^2+n^2$ and $m^2+2n^2=3$, and we may put

$$\left.\begin{aligned} 2m^2+n^2&=3h^2\\ m^2+2n^2&=3k^2\end{aligned}\right\}\ldots\ldots(7).$$

From the first of these, $2(m^2-h^2)=h^2-n^2$, me may now take $\frac{h-n}{2(m-h)}$ in its lowest terms$=\lambda/\mu$, so that

$$2\lambda(m-h)=\mu(h-n),\quad \mu(m+h)=\lambda(h+n),$$

whence $(2\lambda^2+\mu^2)m-(2\lambda^2-\mu^2)h=2\lambda\mu h$, and $m/h=\frac{2\lambda\mu+2\lambda^2-\mu^2}{2\lambda^2+\mu^2}$.

Whence as before,

$$\left.\begin{aligned} h&=2\lambda^2+\mu^2\\ m&=2\lambda\mu+2\lambda^2-\mu^2\\ n&=2\lambda\mu-2\lambda^2+\mu^2\end{aligned}\right\}\ldots\ldots(8),$$

or the halves of these, respectively, which will have the same form, except that m and n are transposed.

Similarly from the equation $m^2+2n^2=3k^2$, we get

$$m^2-k^2=2(k^2-n^2)\text{ and }2\lambda_1(k-n)=\mu_1(m-k).$$

Whence

$$\left.\begin{aligned} k&=2\lambda_1^2+\mu_1^2\\ m&=2\lambda_1\mu_1-2\lambda_1^2+\mu_1^2\\ n&=2\lambda_1\mu_1+2\lambda_1^2-\mu_1^2\end{aligned}\right\}\ldots\ldots(9),$$

or the halves of these, respectively, which will have the same form except that the expressions for m and n will be interchanged.

From the first set $m+n=4\lambda_1\mu_1$, and from the second $m+n=4\lambda_1\mu_1$.

$\therefore\ \lambda\mu=\lambda_1\mu_1$.

Also $2\lambda\mu+2\lambda^2-\mu^2=2\lambda_1\mu_1-2\lambda_1^2+\mu_1^2$,

or $=2\lambda_1\mu_1+2\lambda_1^2-\mu_1^2$.

In the first case, $2\lambda^2-\mu^2=-2\lambda_1^2+\mu_1^2$.

$\therefore\ 2(\lambda^2+\lambda_1^2)=\mu^2+\mu_1^2$,

$2\lambda_1^2(\lambda^2+\lambda_1^2)=\lambda_1^2\mu^2+\lambda_1^2\mu_1^2=\lambda_1^2\mu^2+\lambda^2\mu^2=(\lambda^2+\lambda_1^2)\mu^2$.

$\lambda^2+\lambda_1^2$ cannot be zero.

$\therefore 2\lambda_1^2=\mu^2$, or $\sqrt{2}\lambda_1=\mu$, which is impossible since λ_1 and μ are integers.
We must now therefore try the second case,

$$2\lambda\mu+2\lambda^2-\mu^2=2\lambda_1\mu_1+2\lambda_1^2-\mu_1^2.$$

Whence $2(\lambda^2-\lambda_1^2)=\mu^2-\mu_1^2$,

$$2\lambda_1^2(\lambda^2-\lambda_1^2)=\lambda_1^2\mu^2-\lambda_1^2\mu_1^2=\lambda_1^2\mu^2-\lambda^2\mu^2=(\lambda_1^2-\lambda^2)\mu^2.$$

$\therefore (2\lambda_1^2+\mu^2)(\lambda^2-\lambda_1^2)=0.$
$2\lambda_1^2+\mu^2$ cannot be zero.
$\therefore$ the only solution is $\lambda^2-\lambda_1^2=0$, or $\lambda=\lambda_1$.
Hence from $\lambda\mu=\lambda_1\mu_1$, $\mu=\mu_1$.
Now $h=2\lambda^2+\mu^2=2\lambda_1^2+\mu_1^2=k$ [from (8) and (9)].
$\therefore 2m^2+n^2=m^2+2n^2$ [from (7)].
$\therefore m^2=n^2$.
But m and n have by hypothesis no common factor (greater than unity).
$\therefore m=n=1$.
$\therefore q-p=r-s$ [from (6)], and $pq=rs$.
$\therefore q+p=r+s$, and $q=r$, $p=s$. Also $2q-p=2s-r=2p-q$.
$\therefore q=p$, and as they have no common measure, each$=1$.
$\therefore p=q=r=s=1$, and

$$\left.\begin{aligned} x&=p^2+2q^2-2pq=1\\ y&=4pq-2q^2-p^2=1\end{aligned}\right\}\text{ from (4).}$$

$$\begin{aligned} a&=\sqrt{(2x^2-y^2)}=1\\ b&=\sqrt{(2y^2-x^2)}=1.\end{aligned}$$

(These values of x and y are also derivable from the equations

$$\begin{aligned} x&=4rs-2s^2-r^2\\ y&=2s^2+r^2-2rs,\end{aligned}$$

so that all the equations are consistent.)

$\therefore x=y=a=b=1$ is the only solution in integers of the two equations when x and y are prime to one another.

Of course an infinite number of solutions may be obtained by multiplying x, y, a, and b by a common factor, C, so that

$$\begin{aligned} 2C^2-C^2&=C^2 \text{ for } 2x^2-y^2=a^2,\\ 2C^2-C^2&=C^2 \text{ for } 2y^2-x^2=b^2.\end{aligned}$$

PROBLEMS FOR SOLUTION.

ARITHMETIC.

114. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Does it pay a $4-carpenter using a dozen four-penny nails per minute, to pick up a dropped nail? At this rate, should twenty-penny nails be picked up?

115. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Ia.

Where shall a pole 120 feet high be broken so that the top may rest on the ground 40 feet from the foot? (Solve by arithmetic.)

*** Solutions of these problems should be sent to B. F. Finkel not later than August 10.

ALGEBRA.

102. Proposed by J. MARCUS BOORMAN, Woodmere, N. Y.

Solve $2x+\sqrt{(x^2-7)}=5$. [See *Hind's Algebra*, page 447.]

103. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Given the equation $x^m+p_1x^{m-1}+p_2x^{m-2}+\ldots\ldots p_{m+1}x+p_m=0$ freed from multiple roots. Prove that its discriminant is positive or negative according as the number of pairs of complex roots is even or odd.

*** Solutions of these problems should be sent to J. M. Colaw not later than August 10.

104. Prize Problem. $2.50 for the best solution.

Compute to three decimal places each of the roots of the equation $x^2+y=2$, $x+y^2=6$.

*** Solutions of this problem should be sent to B. F. Finkel not later than September 1.

GEOMETRY.

123. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Ia.

A étant le point d' intersection des médianes d' un triangle ABC, démontrerque $AB^2+BC^2+CA^2=3(GA^2+GB^2+GC^2)$. [Ex. 84, Géométrie. No. 2, 1re Anne *L' Éducation Mathématique.*]

*** Solutions of this problem should be sent to B. F. Finkel not later than August 10.

CALCULUS.

92. Proposed by B. F. SINE, Principal of Capon Bridge Normal School, Capon Bridge, W. Va.

How much wood is taken from a log 12 inches in diameter, by boring a two-inch hole through the center, the axis of hole being perpendicular to axis of log?

93. Proposed by JOHN R. JEFFREY, Student in Ohio State University, Columbus, Ohio.

Solve the following differential equation:

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2x, \text{ when } x<1.$$

94. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Ia.

Find the minimum isosceles triangle that can be described about a given ellipse, having its base parallel to the major axis. [Ex. 16, page 166, *Rice and Johnson's Differential Calculus.*

*** Solutions of these problems should be sent to J. M. Colaw not later than August 10.

MECHANICS.

91. Proposed by CHARLES C. CROSS, Whaleyville, Va.

The bow of a boat which is a inches wide is inclined at an angle α. When in motion in perfectly calm water the water was found to rise b inches on the bow. Required the velocity of the boat.

92. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A particle, starting at the vertex, slides down a smooth parabolic curve. Prove that in order to leave the curve at the extremity of the latus rectum, the initial velocity of the particle must be $pg[\sqrt{2}-1]$ where p is semi-latus rectum.

*** Solutions of these problems should be sent to B. F. Finkel not later than August 10.

AVERAGE AND PROBABILITY.

75. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Find the mean area of all plane rectilineal right triangles having a constant perimeter p.

76. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

In a given ellipse, the extremities of a focal chord are joined with the center. Find the average area of the triangle thus formed.

*** Solutions of these problems should be sent to B. F. Finkel not later than August 10.

MISCELLANEOUS.

79. Proposed by S. HART WRIGHT, A. M., Ph. D., Penn Yan, N. Y.

In latitude 42° 30′ N.$=\lambda$, a tree 100 feet long$=\alpha$, leans in the direction S. 60° W.$=\beta$, with an angle of elevation with the level ground, of 30°$=\gamma$. The sun's declination being 1° 36′ 24″ N.$=\delta$, in what direction will the shadow of the tree point, when the sun is on the meridian?

80. Proposed by SYLVESTER ROBINS, North Branch Depot, N. J.

Exhibit ten initials in that infinite series of integral, rational rhombuses wherein the area of every term is one unit less than the square of its side.

*** Solutions of these problems should be sent to J. M. Colaw not later than August 10.

EDITORIALS.

It should have been stated in our last issue that the excellent portrait of Sophus Lie was furnished us through the kindness of Dr. Halsted.

The University of Wisconsin is offering for the Academic year beginning 1899, a very excellent course of Graduate Study in Electrical Engineering.

We offer again in this issue a prize of $2.50 for the best solution of problem 104 in algebra. Any person under twenty-one years of age is eligible to compete for the prize.

Advanced sheets of a School Algebra, by Drs. Fisher and Schwatt of the University of Pennsylvania have just reached us. A further notice of the book will be given in the June number of the MONTHLY.

Dr. George Bruce Halsted has been invited to present a Report on Progress in non-Euclidean Geometry at the coming Columbus meeting of the American Association for the Advancement of Science. Dr. Halsted has accepted the invitation and will commence soon the preparation of the report, which is to be quite exhaustive.

During the year 1899-1900, Drury College will offer the following electives in Mathematics: Advanced Integral Calculus, Differential Equations, Projective Geometry, Analytical Mechanics, and Theory of Functions.

The Register and Eleventh Official Announcement of Clark University has just reached us. The following are some of the Courses offered in Mathematics for the year 1899-1900: Differential Geometry, Algebraic Invariants, Analytical Geometry of Higher Surfaces and Twisted Curves, Elliptic Functions, Differential equations, and Calculus of Variations, Finite Continuous Groups, and Theory of Numbers.

The following are some of the advanced courses of Mathematics offered for the year 1899-1900 at the University of Chicago: Twisted Curves and Surfaces, Associate Professor Maschke; Projective Geometry, Professor Moore; Theory of Invariants, Professor Bolza; Continuous Groups, Professor Bolza; Theory of Functions of a Complex Variable, Professor Moore and Associate Professor Maschke; Elliptic Functions, Professor Bolza; Hyperelliptic Functions, Professor Bolza; Abstract Groups, Associate Professor Maschke; Elliptic Modular Functions, Professor Moore; Theory of Substitution, Professor Moore; Theory of Numbers, Assistant Professor Young, etc., etc.

Harvard University has just issued its Course of Study in Mathematics for the year 1899-1900. Among the courses offered in advanced Mathematics are the following: General Theory of Surface, Prof. J. M. Peirce; Dynamics of a Rigid Body, Professor Byerly; Quaternions with Applications to Geometry and Mechanics, Prof. J. M. Peirce; Trigonometric Series, Introduction to Spherical Harmonics, Potential Functions, Professors Byerly and B. O. Peirce; Theory of Functions (Second Course), Riemann's Theory of Functions, Professor Osgood; Algebra—Galois's Theory of Equations, Professor Osgood; Lie's Theories as Applied to Differential Equations, Dr. Bouton; etc., etc. The courses offered at Harvard are sufficiently varied and extensive to meet the wants of any student of mathematics.

BOOKS AND PERIODICALS.

A Text-Book of Physics.—Sound. By J. H. Poynting, Sc. D., F. R. S., Late Fellow of Trinity College, Cambridge, Professor of Physics in Mason University College, Birmingham; and J. J. Thomson, M. A., F. R. S., Sc. D., Dublin; Hon. D. L., Princeton; Fellow of Trinity College, Cambridge; Cavendish Professor of Experimental Physics in the University of Cambridge. Royal 8vo. Cloth, 174 pages. Price 8*s*. 6*d*. London: Charles Griffin & Co., and Philadelphia: J. B. Lippincott & Co.

The authors are preparing a Text-book of Physics of which the account of the phenomena of sound and the the theory of these phenomena as presented in this book forms only one part. The presentation of the subject of sound as here given is such as to enable the student who has no knowledge of advanced mathematics to obtain a good understanding of this branch of physics. The book is well printed, the quality of paper used is very good, but the illustrations of some pieces of apparatus are poor. B. F. F.

The Elements of Graphic Statics. A Text-book for Students of Engineering. By L. M. Hoskins, Professor of Applied Mechanics in the Leland Stanford Junior University. Revised Edition. 8vo. Cloth, 200 pages. Price, $2.25. New York: The Macmillan Co.

In this book is found not only that which meets the needs of the student, but also that which is useful to the structural engineer as well. B. F. F.

Röntgen Rays. Memoirs by Röntgen, Stokes, and J. J. Thomson. Translated and edited by George F. Barker, LL. D., Professor of Physics in the University of Pennsylvania. 8vo. Cloth, 76 pages. Price, 60 cents. New York: Harper & Bros.

This is volume III of Harper's Scientific Memoirs, edited by J. S. Ames, Ph. D., Professor of Physics in Johns Hopkins University. This little book contains Professor Röntgen's paper in which he announced to the world his wonderful discovery; a biography of Professor Röntgen; a paper On the Nature of the Röntgen Rays, by Sir G. G. Stokes; a biographical sketch of Stokes; a paper On the Theory of the Connection between Cathode and Röntgen Rays, by J. J. Thomson; and a biographical sketch of Thomson. This little

volume makes very interesting reading for any one interested in one of the most wonderful discoveries of the nineteenth century. B. F. F.

A History of Physics in its Elementary Branches, including the Evolution of Physical Laboratories. By Florian Cajori, Ph. D., Professor of Physics in Colorado College. 8vo. Cloth, 322 pages. Price, $1.60. New York: The Macmillan Co.

This History of Physics is in line of excellence and interest with the History of Mathematics by Dr. Cajori. Such an account of the progress and development of Physics, as well as a short sketch of the investigators who have enlarged and enriched its domain by careful research and discovery, has long been found desirable. Every teacher and student of Physics should read this book, as the history of any science has a stimulating effect and helps to make it attractive. De Morgan said, "No man should think it a waste of time to learn something of the history of his own subject; nor is the investigation of laborious methods now fallen into disuse, or errors once commonly accepted the least valuable of mental disciplines." B. F. F.

The Modern Theory of Solution. Memoirs by Pfeffer, Van't Hoff, Arrhenius, and Raoult. Translated and edited by Harry C. Jones, Ph. D., Associate in Physical Chemistry in Johns Hopkins University. 8vo. Cloth, 128 pages. Price, $1.00. New York: Harper & Bros.

This is the fourth volume of Harper's Scientific Memoirs. It contains Pfeffer's Osmotic Investigations; biography of Pfeffer; Van't Hoff's The Role of Osmotic Pressure in the Analogy between Solutions and Gases; biography of Van't Hoff; Arrhenics's On the Dissociation of Substances Dissolved in Water; biography of Arrhenius; Raoult's The General Law of the Freezing of Solvent and On the Vapor-Pressure of Etheriel Solutions; The General Law of the Vapor-Pressure of Solvents; and a biography of Raoult. B. F. F.

Algebra Elementare Ad Uso Dei Licei e Degli Instituti Fecnici (I° Biennio) Secondo I Programmi Governativi con Copiose Note Storiche Molte Consigli Pratici Per Indirizzare L' Allunno Alla Resoluzione Degli Esercizi Piu' di 2000 Esercizi e Problemi Graduati da Risolvere e Circa 400 Esercizi e Problemi Minutaments Risolti. Pages 428.

This is an elementary algebra of unusual value, as the definition of all algebraic operations are given according to the advanced notions of modern mathematics. The numerous illustrative problems are worked out in detail, the historical notes are of value, and the many problems of the various chapters well chosen. The work throughout bears evidence of the painstaking care of the author in its preparation. B. F. F.

The following periodicals have been received: *Journal de Mathématiques Élémentaires*, (1er Mai 1899); *The American Journal of Mathematics*, (April, 1899); *L'Intermédiaire des Mathématiciens*, (April 1899); *Bulletin of the American Mathematical Society*, (May, 1899); *The Kansas University Quarterly*, (January, 1899); *The Monist*, (April, 1899); *The Ohio Teacher*, (April, 1899); *The Educational Times*, (May 1, 1899); *The Mathematical Gazette*, (February, 1899).

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited and published by John Brisben Walker. Price, $1.00 per year in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

THE AMERICAN MATHEMATICAL MONTHLY.

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ON SYMMETRIC FUNCTIONS.

By E. D. ROE, Jr., Associate Professor of Mathematics in Oberlin College.

[Concluded from May Number.]

H. APPLICATION OF THE THEORY TO COMPUTATION.

The limits of the present article do not permit of thoroughgoing application in the calculation of coefficients of terms of symmetric functions. The writer has used the present theory and the formula of G, 2, in such calculations, and by means of the parallel theory for the resultant, and the equivalent formula of A, 4, he has calculated all the coefficients of the normal forms of all resultants up to and including the resultant $R_{5,4}$.* By means of these methods the calculation is very easy. The advantages of the theory and formulas here developed in the calculation of tables of symmetric functions may be stated to be as follows: Not only is the symmetry of the table established by the fundamental relations, whereby half the coefficients are repeated, but also by the same relations the numerical equality of certain other coefficients in the same table and of others in different tables is immediately established. In addition to this many coefficients of a table which are not normal forms are easily reduced to such as are normal forms of a lower table and have been previously calculated. Also the coefficients of the completely reducible forms, which have the form of the general formula obtained for them in F, 2, are the values of $(-1)^n$. To this may be added certain coefficients whose value is zero by the fundamental conditions. [We might

*The results are published in the before-mentioned thesis.

also add the general formula, for the normal forms where $n=2$, which can be proved without difficulty, viz :

$$\begin{pmatrix} 0^{\lambda_0} & 1^{\lambda_1} & 2^{\lambda_2} \\ 0^m & m & 0 \end{pmatrix} = (-1)^{\lambda_1+\lambda_2} \frac{m(\lambda_1+\lambda_2-1)!}{\lambda_1!\,\lambda_2!}].$$

When all these means of obtaining coefficients have been applied, the actual number of normal forms in a table requiring calculation is comparatively very small, and by means of the formula of F, 2, the calculation is easily made, giving the coefficient at once as the sum of earlier calculated coefficients.

NOTE ON ELIMINATION BY MEANS OF SYMMETRIC FUNCTIONS.

In behalf of the extension of the method given under I, B, the following details may be added :

1. On the Resolution of Aronhold's Operator into Three Operators.

The term containing $(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m}$ in $\delta R_{m,n}$ must have come from $a_i(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m} \mid i0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m} \mid$ by the use of $b_iD_{a_i}$ in as many ways as there are operators of this kind when i takes all values from $i=0$, to $i=n$. $b_iD_{a_i}$ applied to $a_i(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m} \mid i0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m} \mid$ gives

$$(\lambda_i+1)b_i \mid i0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m} \mid (a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m}$$

$$=(\lambda_i+1)b_i \mid 0^{\lambda_0}1^{\lambda_1}\ldots.i^{\lambda_i+1}\ldots.m^{\lambda_m} \mid (a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m},$$

and the coefficient of $(a_0)^{\lambda_0}(a_1)^{\lambda_1}\ldots.(a_m)^{\lambda_m}$ in $\delta R_{m,n}$ is

$$\sum_{i=0}^{i=n} (\lambda_i+1)b_i \mid i0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m} \mid \equiv 0.$$

Thus Aronhold's operator is resolved into three operators :

(1). 0, 1, 2, ...n *applied to* $\mid 0^{\lambda_0}1^{\lambda_1}\ldots.m^{\lambda_m} \mid$.

(2). *The literal operators* b_0, b_1, ... b_n *applied to the preceding.*

(3). *The numerical operators* λ_0+1, λ_1+1, ...λ_n+n.

Here λ_i+1 associated with b_i is the exponent of i in the associated stroked form which results from the first operation. Of course $\lambda_0+\lambda_1+\ldots.\lambda_m=n-1$, and there will be

$$\frac{m(m+1)\ldots.(m+n-1)}{1.2\ldots.(n-1)}=e,$$

(the number of homogeneous products of m elements to $n-1$ dimensions) such identical equations of the form